THE LEAST GRADIENT PROBLEM IN THE FREE MATERIAL DESIGN

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Stating the least gradient problem

A version of the Free Material Design maybe stated as follows: given region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ a load at the boundary consistent with the equilibrium, i.e. $\int_{\partial\Omega} g\, dS = 0$ find the optimal distribution p of the material. By optimality we mean that

$$(1) \qquad \int_\Omega |p| = \inf\{\int_\Omega |q| :\ q \in L^1(\Omega, \mathbb{R}^d),\ \operatorname{div} q = 0,\ q \cdot \nu|_{\partial\Omega} = g\}.$$

Here, ν is the outer normal to $\partial\Omega$. It obvious from the statement of (1) that one should expect to find a solution in the space of Radon measures, $\mathcal{M}$, on Ω.

One can look for a dimension reduction of (1), which is simple, when $d = 2$. We notice that (1) is equivalent to

$$(2) \qquad \int_\Omega |Du| = \inf\{\int_\Omega |Dv| :\ v \in BV(\Omega),\ v|_{\partial\Omega} = f\},$$

where $BV(\Omega)$ is space of functions with bounded total variation and $\frac{\partial f}{\partial \tau} = g$ and τ is a tangent vector to $\partial\Omega$. The equivalence is given by the mapping $BV(\Omega) \ni u \mapsto QDu \in \mathcal{M}$, where Q is the rotation by $\frac{\pi}{2}$, for details see [3].

Existence of solution in strictly convex domains for different boundary conditions

It is well-known fact that if $f \in C(\partial\Omega)$ and $\Omega \subset \mathbb{R}^2$ is strictly convex, then there exists a unique solution to (2), see [5]. For more general data neither existence, nor uniqueness is obvious. A part of the problem is that the problem (2) is ill-posed, because the following functional $\Phi : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$, given by $\Phi(u) = \int_\Omega |Du|$, if and only if $u \in BV(\Omega)$ and $u|_{\partial\Omega} = f$, otherwise $\Phi(u) = +\infty$, is not lower semicontinuous. Nonetheless, we can show

Theorem 1. (see [2], [3])
If $\Omega \subset \mathbb{R}^2$ is strictly convex, $f \in BV(\partial\Omega)$, then problem (2) has at least one solution. □

Here is an **Example** of a solution, [3]. If $\partial\Omega$ is parametrized by arclength, $[0, L) \ni s \mapsto x(s) \in \partial\Omega$, then we take $f = (\alpha_1 + \alpha_2)\chi_{[s_2,s_2)} + \chi_{[s_2,L)}$, $s \in [s_2, L)$. The solution, u, takes three values, 0, α_1, $\alpha_1 + \alpha_2$ and it is depicted on Fig. 1.

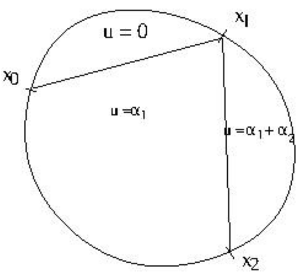

Fig. 1

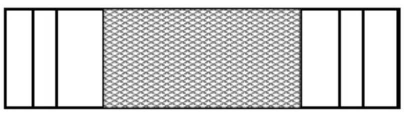

Fig. 2

By modifying the method of [5] we can show existence of solution to (2) when continuous data are specified only on $\Gamma \subsetneq \Omega$.

Theorem 2. (see [3])
If $\Omega \subset \mathbb{R}^2$ is strictly convex, $\Gamma \subsetneq \Omega$ is a smooth arc, $f \in C(\bar{\Gamma})$, then problem (2), when $u|_\Gamma = f$ is in place of $u|_{\partial\Omega} = f$, has at least one solution. □

Existence of solutions in convex but not strictly convex domains

The main problem for existence is presence of nontrivial line segments ℓ in the boundary of Ω, we call them *flat parts*. We shall say that a continuous function $f \in C(\partial\Omega)$ *satisfies the admissibility condition #1* on a flat part ℓ iff f restricted to ℓ is monotone.

We associate with f on a flat piece of the boundary, ℓ, a family of closed intervals $\{I_i\}_{i\in\mathcal{I}}$ such that $I_i = [a_i, b_i]$ is contained in the interior of ℓ relative to $\partial\Omega$. On each I_i function f attains a local maximum or minimum on each ℓ and each I_i is maximal with this property. We also set $e_i = f(I_i)$, $i \in \mathcal{I}$. For the sake of making the notation concise we will call I_i a *hump*.

After this preparation we state the admissibility condition for non-monotone functions. A continuous function f, which is not monotone on a flat part ℓ, *satisfies the admissibility condition #2* iff for each hump $I_i = [a_i, b_i] \subset \ell$ and $e_i := f([a_i, b_i])$, $i \in \mathcal{I}$ the following inequality holds,

$$\text{dist}\,(a_i, f^{-1}(e_i) \cap (\partial\Omega \setminus I_i)) + \text{dist}\,(b_i, f^{-1}(e_i) \cap (\partial\Omega \setminus I_i)) < |a_i - b_i|. \tag{3}$$

Theorem 3. (see [4])
Let us suppose that Ω is convex and $f \in C(\partial\Omega)$. In addition, $\partial\Omega$ has a finite number of flat parts $\{\ell_k\}_{k=1}^N$. If f satisfies the admissibility conditions #1 or #2 on each flat part $\{\ell_k\}_{k=1}^N$ of $\partial\Omega$, then there is a continuous solution to the least gradient problem. □

We can extend this result also to the case $f \in BV(\partial\Omega)$ or an infinite number of flat parts of $\partial\Omega$.

Example
We define $\Omega = (-L, L) \times (-1, 1)$, $L > 2$. We take, $f_i \in C(\partial\Omega)$, $i = 1, 2$ given by $f_1(x, y) = \cos(\frac{\pi}{2} y)$ and $f_2(x, y) = \cos(\frac{\pi}{2} \frac{x}{L})\chi_{|x|>L-2}(x) + \chi_{|x|\le L-2}(x)$. For f_1 problem (2) has no solution, while for f_2 there is a unique solution whose level sets are shown on Fig. 2. The shaded area is a level set of positive Lebesgue measure.

We also discuss the lack of uniqueness of solutions. We show that non-uniqueness of solutions to (2) is related to level sets of u with positive 2-d Lebesgue measure and discontinuities of f. This is done in [1].

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