ON THE FREE MATERIAL DESIGN FOR THE MULTIPLE LOADING CONDITIONS

S. Czarnecki, R. Czubacki and T. Lewiński

Warsaw University of Technology, Faculty of Civil Engineering, Warsaw, Poland
e-mail: t.lewinski@il.pw.edu.pl

1. The stress-based versions of the free material design: AMD, CMD, IMD and YMD

The free material design method (FMD) is one of the methods of optimum design of structural topology. The present paper discusses three variants of FMD.

In the AMD method (*anisotropy material design*, see [1], [2]) all components C_{ijkl} of Hooke's tensor $\mathbf{C}$ are design variables; $\operatorname{tr}\mathbf{C} = C_{ijij}$ represents the unit cost of the design within the given spatial design domain Ω.

In the CMD method (*cubic material design*, [3]) the principal elastic moduli *a,b,c* and the triplet of unit vectors **n**, **m**, **p** (the notation of Walpole [4] being used) are design variables. The unit cost is assumed as $\operatorname{tr}\mathbf{C} = a + 3b + 2c$.

In the IMD method (*isotropic material design*, see [5,6,7]) the bulk modulus *k* and the shear modulus μ are design variables; the unit cost is assumed as $\operatorname{tr}\mathbf{C} = 3k + 10\mu$.

In the YMD (*Young modulus design*, see [8]) the Young modulus is the only design variable, the Poisson ratio being kept fixed. The unit cost is proportional to the Young modulus.

Consider *n* variants of traction loads. The values of linear forms $f^{\alpha}(\mathbf{v}),\ \alpha = 1,...,n$, on a virtual displacement field **v** represent the values of the virtual work of the loads for the subsequent load variants. Let $\mathbf{u}^{\alpha}$ represent the displacement field caused by the α th load. In order to construct the Pareto front we consider the problem of minimization of the functional

$$\wp_{\eta} = \eta_1 f^1(\mathbf{u}^1) + ... + \eta_n f^n(\mathbf{u}^n), \quad \eta_1 + ... + \eta_n = 1 \tag{1}$$

being a convex combination of the compliances corresponding to the subsequent load variants. The aim of the methods AMD, CMD, IMD and YMD is minimization of the functional (1) over the design variables satisfying the cost condition: the integral of the unit cost is predefined and equals Λ. Each of the optimization methods discussed leads to an auxiliary problem of the form

$$\min\left\{ \int_{\Omega} \rho\left(\sqrt{\eta_1}\boldsymbol{\tau}^1,...,\sqrt{\eta_n}\boldsymbol{\tau}^n\right)dx \;\middle|\; \boldsymbol{\tau}^{\alpha} \in \Sigma_{\alpha}(\Omega),\ \alpha = 1,...,n \right\} \tag{2}$$

where $\Sigma_{\alpha}(\Omega)$ is the set of statically admissible stress fields $\boldsymbol{\tau}^{\alpha} = (\tau_{ij}^{\alpha})$ corresponding to the α th load variant, while $\rho(\boldsymbol{\sigma}^1,...,\boldsymbol{\sigma}^n)$ is a *gauge function* of *n* stress fields, characteristic for the AMD, CMD, IMD and YMD methods, respectively. These gauge functions corresponding to the optimization methods discussed are norms and in general the minimizer of (2) is a Radon measure, see [9], where similar problems are discussed. Having found the minimizer of (2) one can determine the optimal moduli. Within the approaches: AMD, IMD and YMD we have at our disposal explicit formulae for making this computation efficiently. The formulae for the optimal moduli of cubic symmetry are known only for *n*=1, see [3]. One of the aims of the present paper is to derive the formulae of the method CMD for *n*>1.

2. The kinematic approach

The auxiliary problem (2) can be rearranged to its dual form involving virtual (or adjoint) displacements:

$$\max\left\{ f^1\left(\sqrt{\eta_1}\mathbf{v}^1\right)+...+f^n\left(\sqrt{\eta_n}\mathbf{v}^n\right) \;\middle|\; \mathbf{v}^\alpha \in V(\Omega) \text{ and } \rho^*\left(\boldsymbol{\varepsilon}(\mathbf{v}^1),...,\boldsymbol{\varepsilon}(\mathbf{v}^n)\right)\le 1 \text{ a.e. in } \Omega\right\} \qquad (3)$$

where $\boldsymbol{\varepsilon}(\mathbf{v})$ is the symmetric part of the gradient of the field **v**. For the methods: IMD and YMD the dual norms $\rho^*(.)$ have been derived in [7,8]. The aim of the present paper is to write down (3) in an explicit form for the AMD and CMD settings.

The main property of the dual pair of problems (2), (3) is that the solutions may vanish on an essential part of the design domain. The effective domain of the solutions determines the material subdomain where the structure emerges, capable of transmitting the given loads to the given supporting edge.

3. Final remarks

In the case of a single load condition the free material design methods in its anisotropic version AMD leads to the optimal, but highly degenerated Hooke tensor characterized by a single non-vanishing eigenvalue, hence the zero eigenvalue is characterized by the multiplicity 5. The CMD method leads to the less degenerated Hooke tensor in which the zero eigenvalue has multiplicity 3. By admitting more than one load condition the optimum designs may become non-singular, if the load variants are appropriately chosen. In case of the AMD method at least six load conditions are necessary to make the Hooke tensor positive definite. The aim of the paper is to clear up this question in the context of CMD.

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